Journal of Engineering Physics and Thermophysics, Vol. 73, No. 5, 2000

METHOD OF PAIRED INTEGRAL EQUATIONS WITH *L*-PARAMETER IN PROBLEMS OF NONSTATIONARY HEAT CONDUCTION WITH MIXED BOUNDARY CONDITIONS FOR AN INFINITE PLATE

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UDC 517.968,536.24

For solving a nonstationary equation of heat-conduction in cylindrical coordinates with mixed discontinuous boundary conditions prescribed on one of the surfaces ($z = 0$) of an infinite plate, a method of paired integral equations with L-parameter is used. Moreover, on the other surface ($z = h$) of the plate the boundary conditions are prescribed unmixed.

It is required to solve the heat-condition equation for the excess-temperature function:

$$\theta_{rr}(r, z, \tau) + \frac{1}{r}\theta_r(r, z, \tau) + \theta_{zz}(r, z, \tau) = \frac{1}{a}\theta_\tau(r, z, \tau),$$
$$r > 0, \ \tau > 0, \ 0 < z < h, \tag{1}$$

where r and z are the cylindrical coordinates, τ is the time, $a > 0$ is the thermal diffusivity coefficient, $\theta(r, z, \tau) = T(r, z, \tau) - T_0$, and T_0 = const is the initial temperature of the plate.

Prescribed are the initial condition

$$\theta(r, z, 0) = T(r, z, 0) - T_0 = 0, \tag{2}$$

a homogeneous boundary condition at $z = h$

$$\theta(r, h, \tau) = 0 \tag{3}$$

and mixed boundary conditions at $z = 0$

$$-\theta_z(r, 0, \tau) = \frac{q^*(r, \tau)}{\lambda} = q(r, \tau), \ \ 0 < r < R, \tag{4}$$

$$\theta(r, 0, \tau) = 0, \ \ R < r < \infty, \tag{5}$$

where $\lambda > 0$ is the thermal conductivity coefficient.

Applying the Hankel–Laplace transformation to problem (1)-(5) of the form

$$\bar{\theta}_H(p, z, s) = \int_0^\infty \int_0^\infty \theta(r, z, \tau) \exp(-s\tau) J_0(pr) r dr d\tau, \ \ \text{Re}\, s > 0, \tag{6}$$

where $J_0(pr)$ is the Bessel function of the first kind and zero order, and taking into account the boundedness of the temperature $\theta(r, z, \tau)$ on the axis $r = 0$ and for $r \to \infty$, the solution of the heat-conduction equation (1) in the region of L-transforms can be written [1] in the form

Belarusian State University, Minsk, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 73, No. 5, pp. 902-906, September–October, 2000. Original article submitted February 10, 2000.

$$\bar{\theta}(r, z, s) = \bar{T}(r, z, s) - \bar{T}_0 = \int_0^{\infty} \bar{A}(p, s) \frac{\sinh\left[(h - z)\sqrt{\left(p^2 + \frac{s}{a}\right)}\right]}{\sinh\left(h\sqrt{\left(p^2 + \frac{s}{a}\right)}\right)} J_0(pr)\, dp, \tag{7}$$

where $\bar{A}(p, s)$ is the unknown analytical function. Here and henceforth it is assumed that Re $s > 0$, and, for brevity, this is omitted in the representation.

After application of the L-transformation to conditions (4) and (5) the mixed boundary conditions for $z = 0$ will take the form

$$-\bar{\theta}_z(r, 0, s) = \bar{q}(r, s), \quad 0 < r < R, \tag{8}$$

$$\bar{\theta}(r, 0, s) = 0, \quad R < r < \infty. \tag{9}$$

Assuming in formula (7) $z = 0$ and taking into account conditions (8) and (9), it is possible to pass to paired integral equations with the L-parameter:

$$\int_0^{\infty} \bar{A}(p, s)\sqrt{\left(p^2 + \frac{s}{a}\right)} \cot\left(h\sqrt{\left(p^2 + \frac{s}{a}\right)}\right) J_0(pr)\, dp = \bar{q}(r, s), \quad 0 < r < R, \tag{10}$$

$$\int_0^{\infty} \bar{A}(p, s)\, J_0(pr)\, dp = 0, \quad R < r < \infty, \tag{11}$$

from which it is necessary to determine the analytical function $\bar{A}(p, s)$.

To solve the paired integral equations (10) and (11), we introduce the new analytical function $\bar{\varphi}(t, s)$ with the aid of the relation [1]

$$\bar{A}(p, s) = \frac{p}{\sqrt{\left(p^2 + \frac{s}{a}\right)}} \int_0^{R} \bar{\varphi}(t, s) \sin\left(t\sqrt{\left(p^2 + \frac{s}{a}\right)}\right) dt. \tag{12}$$

On substitution of (12) into the second paired equation (11), we can easily verify that Eq. (11) is readily satisfied according to the value of the discontinuous integral

$$\int_0^{\infty} \frac{pJ_0(pr)}{\sqrt{\left(p^2 + \frac{s}{a}\right)}} \sin\left(t\sqrt{\left(p^2 + \frac{s}{a}\right)}\right) dp = \begin{cases} 0, & 0 < t < r, \\ \dfrac{\cos\left(\left(\frac{s}{a}(t^2 - r^2)\right)^{1/2}\right)}{\sqrt{t^2 - r^2}}, & 0 < r < t. \end{cases} \tag{13}$$

Substituting (12) into the first paired equation (10), we arrive at the integral equation with L-parameter for determining the unknown $\varphi(r, s)$:

$$\int_0^r \frac{t\bar{\varphi}(t,s)}{\sqrt{r^2-t^2}} \exp\left(-\sqrt{\left(\frac{s}{a}(r^2-t^2)\right)}\right) dt - \int_r^R \frac{t\bar{\varphi}(t,s)}{\sqrt{t^2-r^2}} \sin\left(\sqrt{\left(\frac{s}{a}(t^2-r^2)\right)}\right) dt +$$

$$+ \int_0^R \bar{\varphi}(t,s) \sin\left(t\sqrt{\left(\frac{s}{a}\right)}\right) dt + r\int_0^R \bar{\varphi}(t,s)\, dt \int_0^\infty \frac{\exp\left(-h\sqrt{\left(p^2+\frac{s}{a}\right)}\right)}{\sinh\left(h\sqrt{\left(p^2+\frac{s}{a}\right)}\right)} \times$$

$$\times \sin\left(t\sqrt{\left(p^2+\frac{s}{a}\right)}\right) J_1(pr)\, dp = \int_0^r \bar{q}(\rho,s)\,\rho d\rho\,, \quad 0<r<R\,. \tag{14}$$

Equation (14) is the analytical basis for finding the unknown function $\bar{\varphi}(r, s)$, but it is not very convenient for solution. Therefore, we reduce (14) to an integral equation similar to the Fredholm equation, but with the L-parameter. For this purpose, replacing r by μ in Eq. (14), we multiply the left- and right-hand sides of the equation by the integrating factor $\cos\left(\sqrt{\frac{s}{a}(r^2-\mu^2)}\right)2\mu/\sqrt{(r^2-\mu^2)}$ and then integrate the resulting equation for μ going from zero to r. As a result we have

$$\bar{\varphi}(r,s) - \frac{1}{\pi}\int_0^R \bar{\varphi}(\rho,s)\bar{K}(r,\rho,s)\, d\rho = \bar{F}(r,s)\,, \quad 0<r<R\,, \tag{15}$$

where

$$\bar{K}(r,\rho,s) = \frac{\sin\left(\sqrt{\left(\frac{s}{a}\right)}(\rho-r)\right)}{\rho-r} - \frac{\sin\left(\sqrt{\left(\frac{s}{a}\right)}(\rho+r)\right)}{\rho+r} -$$

$$-\frac{1}{h}\int_{h\sqrt{\frac{s}{a}}}^\infty \frac{\exp(-x)}{\sinh(x)}\left[\cos\left(\frac{\rho-r}{h}x\right) - \cos\left(\frac{\rho+r}{h}x\right)\right] dx\,;$$

$$\bar{F}(r,s) = \frac{2}{\pi\lambda}\int_0^r \frac{\bar{q}(\mu,s)\cos\left(\sqrt{\left(\frac{s}{a}(r^2-\mu^2)\right)}\right)\mu}{\sqrt{r^2-\mu^2}}\, d\mu\,.$$

We note that for $h \to \infty$ we directly obtain a solution in the region of L-transforms from formula (7), the paired integral equations with the L-parameter from formulas (10) and (11), and the corresponding integral equations for $\varphi(t, s)$ from Eqs. (14) and (15) for an isotropic semispace with the mixed boundary conditions (8) and (9) at $z = 0$.

Thus, the problem set is actually solved. The main difficulty of calculating the corresponding temperature fields in an infinite plate with the mixed boundary conditions (4) and (5) and unmixed condition (3) consists of the determination [2] of the analytical function $\varphi(r, s)$ from Eq. (15).

In local means of heating the body surface through a circular region $0<r<R$, $z = 0$ the heat-flux density $\bar{q}^*(r, s) = L[q^*(r, \tau)]$ can be represented in the form of the product $\bar{q}^*(r, s) = W(s)q(r)$, where $W(s) =$

$L[W(\tau)]$ is the transform of the specific power of the heat flux that for the original function $W(\tau)$ depends only on time, and $q(r)$ is the distribution of the dimensionless heat flux along the cylindrical coordinate r in the circular region $0 < r < R$.

Then Eq. (15) can be written in the form

$$d\bar{\varphi}^*(r,s) - \frac{1}{\pi}\int_0^R \bar{\varphi}^*(\mu,s)\,\bar{K}(r,\mu,s)\,d\mu = \frac{2}{\pi\lambda s}\int_0^r \frac{q(\mu)\cos\left(\sqrt{\left(\frac{s}{a}(r^2-\mu^2)\right)}\right)\mu}{\sqrt{r^2-\mu^2}}\,d\mu\,, 0 < r < R\,, \tag{16}$$

where

$$\bar{\varphi}^*(r,s) = \frac{\bar{\varphi}(r,s)}{s\bar{W}(s)}\,,\quad \mathrm{Re}\, s\bar{W}(s) > 0\,. \tag{17}$$

Next, we represent the analytical function $\bar{\varphi}^*(r, s)$ in the form of a series [3]:

$$\bar{\varphi}^*(r,s) = \exp\left(-R\sqrt{\left(\frac{s}{a}\right)}\right)\sum_{n=0}^{\infty} \varphi_n(r)\,(\sqrt{s}\,)^{n-2}\,; \tag{18}$$

we substitute this expression into Eq. (16), expand the kernel $\bar{K}(r, \mu, s)$ and the well-known function on the right-hand side of Eq. (16) into the corresponding series, and perform operations of multiplication of the series obtained. As a result, we come to the following equation:

$$\sum_{n=0}^{\infty} \varphi_n(r)\,(\sqrt{s}\,)^{n-2} = \frac{2}{\pi\lambda}\sum_{n=0}^{\infty}\sum_{m=0}^{n} (\sqrt{s}\,)^{n-2} A_{n,m}\int_0^r q(\mu)\left(\sqrt{r^2-\mu^2}\right)^{m-1}\mu d\mu -$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n} (\sqrt{s}\,)^{n-2}\int_0^R C_m(\mu,r)\,\varphi_{n-m}(\mu)\,d\mu -$$

$$-\frac{1}{h}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k} (\sqrt{s}\,)^{n+k-2} F_{k-l}\int_0^R \frac{16h^4\,[mD_{m,l}(\mu,r) - E_{m,l}(\mu,r)]\,\varphi_n(\mu)}{[4h^2m^2 + (\mu+r)^2]\,[4h^2m^2 + (\mu-r)^2]}\,d\mu\,, 0 < r < R\,, \tag{19}$$

where

$$A_{n,m} = \frac{\cos\left(m\frac{\pi}{2}\right) R^{n-m}}{m\,!\,(n-m)\,!\,(\sqrt{a}\,)^n}\,;\quad F_{k-l} = \frac{(-1)^{k-l}}{(k-l)\,!}\left(\frac{2mh}{\sqrt{a}}\right)^{k-l}\,;$$

$$C_m(\mu,r) = \frac{\sin\left(m\frac{\pi}{2}\right)}{m\,!\,(\sqrt{a}\,)^m}\,[(\mu+r)^{m-1} - (\mu-r)^{m-1}]\,;$$

$$D_{m,l}(\mu,r) = \frac{\cos\left(l\frac{\pi}{2}\right)}{4h^2\,l\,!\,(\sqrt{a}\,)^l}\left\{[4h^2m^2 + (\mu+r)^2]\times\right.$$

$$\times (\mu - r)^l - [4h^2m^2 + (\mu - r)^2] (\mu + r)^l \Big\} ;$$

$$E_{m,l}(\mu, r) = \frac{\sin\left(l\frac{\pi}{2}\right)}{8h^3\, l!\, (\sqrt{a})^l} \Big\{[4h^2m^2 + (\mu - r)^2] \times$$

$$\times (\mu + r)^{l+1} - [4h^2m^2 + (\mu + r)^2] (\mu - r)^{l+1}\Big\} .$$

Thus, at $n = 0$, from Eq. (19) we can write the Fredholm integral equation of the second kind to determine $\Phi_0(r)$:

$$\varphi_0(r) = \frac{2}{\pi\lambda} \int_0^r \frac{q(\mu)\,\mu}{\sqrt{r^2 - \mu^2}}\, d\mu - \frac{r}{h^3} \int_0^R \varphi_0(\mu)\,\mu \times$$

$$\times \sum_{m=1}^{\infty} \frac{16h^4 m}{[4h^2m^2 + (\mu + r)^2]\,[4h^2m^2 + (\mu - r)^2]}\, d\mu .$$

The remaining necessary values of $\varphi_i(r)$, $i = 1, 2$, are also determined from formula (19) on equating the terms at the same powers of $\sqrt{s}$ on the left-hand and right-hand sides.

Substituting $\varphi_n(r)$ into series (18), we find the values of $\varphi^*(r, s)$ and, consequently, from formula (17) we determine the values of $\varphi(r, s)$. Further, using $\varphi(r, s)$ in formula (12), we find the value of $A(p, s)$ and then also the transform of the unknown temperature from formula (7). Finally, applying the Laplace integral inversion formula, we can find directly the original function of the excess temperature $\theta(r, z, \tau) = L^{-1}[\theta(r, z, s)]$.

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